Arithmetic properties of vector-valued Siegel modular forms (Analytic and Arithmetic Theory of Automorphic Forms)

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数理解析研究所講究録 第2100巻 2019年 214-221

Arithmetic properties of vector-valued Siegel modular forms

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Motivated by a problem concerning congruences of vector-valued Siegel modular forms, we aim at showing that (at least for sufficiently large scalar weight) vector-valued modular forms always arise from scalar-valued forms on groups of higher rank (more precisely, from Siegel Eisenstein series) . From this we will also get results on the arithmetic nature of Fourier coefficients of vector-valued modular forms. The main idea is similar to the one in Garrett's paper [7]: Using the doubling method, one shows how to use well-known properties of Fourier coefficients of Siegel Eisenstein series of higher rank. The new point here is how to use differential operators of the type investigated by Ibukiyama (see e.g. [9]) to extend everything to the vector-valued case and to emphasize integrality properties.

1 Generalities

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n with the usual action of the group $Sp(n,\mathbb{R})$, given by $(M,Z)\longmapsto M<Z>:=(AZ+B)(CZ+D)^{-1}$. For a polynomial representation $\rho: GL(n,\mathbb{C})\longrightarrow Aut(V)$ on a finite-dimensional vector space $V=V_\rho$ we define an action of $Sp(n,\mathbb{R})$ on V-valued functions on $\mathbb{H}_n$ by

$$(f,M)\longmapsto (f\mid_\rho M)(Z)=\rho(CZ+D)^{-1}f(M<Z>).$$

We choose the smallest nonnegative integer k such that $\rho=\det^k\otimes\rho_0$ with ρ_0 is still polynomial and we call this k the weight of ρ; if ρ itself is scalar-valued, we often write k instead of $\det^k$. For a congruence subgroup Γ of $\Gamma^n:=Sp(n,\mathbb{Z})$ we define $M^n_\rho(\Gamma)$ as the space of Siegel modular forms for ρ w.r.t.Γ, i.e. the set of all holomorphic functions $F:\mathbb{H}_n\longrightarrow V$ satisfying $F\mid_\rho M=F$ for all $M\in\Gamma$; in the case $n=1$ the usual condition in cusps must be added. The subspace of cusp forms will be denoted by $S^n_\rho(\Gamma)$.
For purposes of congruences it is convenient to realize $V=V_\rho$ as $\mathbb{C}^m$ in such a way that

$$\rho(GL(n,\mathbb{Z}))\subset GL(m,\mathbb{Z}). \qquad (1)$$

This is always possible[1]) and we will consider this realization throughout. The Fourier expansion of F is then of type

$$F(Z) = \sum_T a_F(T) e^{2\pi i trace(TZ)},$$

where the Fourier coefficients $a_F(T)$ are in $\mathbb{C}^m$ and T runs over the set $\Lambda^n_{\geq}$ of all symmetric half-integral matrices of size n, which are positive-semidefinite. It makes sense then to define integral modular forms by integrality of all components of all Fourier coefficients:

$$M^n_\rho(\mathbb{Z}) := \{F = \sum_T a_F(T) q^T \in M^n_\rho \mid \forall T : a(T) \in \mathbb{Z}^m\}.$$

Remark: The condition (1) assures that the integrality of $a_F(T)$ depends only on the $GL(n,\mathbb{Z})$-equivalence class of T.

It is natural to ask, for which ρ does

$$M^n_\rho(\mathbb{Z}) \otimes \mathbb{C} = M^n_\rho \tag{2}$$

hold. For scalar-valued ρ it is a well-known statement ; we can also ask similar questions for any congruence subgroup.
The aim of our work is to show that (2) always holds true. To simplify our exposition, we will only consider the case of cusp forms of weight $k > 2n$ here, but we emphasize that our method allows to treat noncuspidal forms as well and to include small weights; also we can extend everyting to congruence subgroups (substituting $\mathbb{Z}$ by the ring of integers in a cyclotomic field if necessary).
Remark: The basic method is to use properties of Siegel Eisenstein series of degree 2n, in particular to use the known rationality and integrality properties of their Fourier coefficients together with the pullback formula. This in not really new: In fact, Garrett [7] employed this method to prove algebraicity properties in the scalar-valued case. Our point is that this method can be used for integrality as well, including the vector-valued case. We emphasize that we deliberately avoid any use of theta series here (the solution of the basis problem using theta series would provide another proof of (2), see eg. [4] for groups $\Gamma_0(N)$ with N squarefree), but we want to use methods applicable to arbitrary Γ).

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[1]I thank Y.Hironaka and G.Nebe, who both indicated that the requested property may be hidden in [8]

2 Construction of integral cusp forms of degree n from Eisenstein series of degree 2n

We start from an (even) weight $k > 2n$ and consider an automorphy factor $\rho = \rho_0 \otimes det^{k'} : GL(n,\mathbb{C}) \longrightarrow GL(V)$ with $k' > k$. Following Ibukiyama [9], there exists a $V \otimes V$-valued holomorphic differential operator $\mathbb{D}$ on $\mathbb{H}_{2n}$, which is a polynomial in derivatives $\frac{\partial}{\partial z_{ij}}$, evaluated on $\mathbb{H}_n \times \mathbb{H}_n \hookrightarrow \mathbb{H}_{2n}$ with equivariance property

$$\mathbb{D}(F\mid_k \iota(M,M')) = \mathbb{D}(F)\mid_\rho^z M \mid_\rho^w M',$$

valid for all C^∞-functions F on $\mathbb{H}_{2n}$ and all $M, M' \in Sp(n,\mathbb{R})$. Here ι denotes the diagonal embedding of $Sp(n) \times Sp(n)$ into $Sp(2n)$, defined by

$$\iota(\begin{pmatrix} a & b \\ c & d \end{pmatrix})(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) := \begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ c & 0 & d & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}$$

and the upper indices z and w indicate that M (M' respectively) have to be applied with respect to the variable z (and w respectively), embedded in $\mathbb{H}_{2n}$ as $(z,w) \hookrightarrow \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$.

We shall apply $\mathbb{D}$ to Siegel's Eisenstein series of degree $2n$, defined by

$$E_k^{2n}(Z) := \sum_{\sim\backslash\begin{pmatrix} * & * \\ C & D \end{pmatrix}} \det(CZ+D)^{-k} = \sum_{\mathcal{T}} b_k^{2n}(\mathcal{T}) e^{2\pi i trace(\mathcal{T}\cdot Z)}$$

The arithmetic nature of the Fourier coefficients $b_k^{2n}(\mathcal{T})$ is well explored by Siegel, Kitaoka, Katsurada, Shimura and others. In particular, the Fourier coefficients are rational with *bounded* denominators.

To describe the Fourier expansions of $\mathbb{D}E_k^{2n}$, we introduce a $V \otimes V$-valued polynomial $\mathcal{P}$ defined on symmetric matrices $\mathcal{P}$ of size $2n$ by

$$\mathbb{D}\,\mathfrak{e}_{\mathcal{T}}(z,w) = \mathcal{P}(\mathcal{T}) e^{2\pi i trace(Tz+Sw)},$$

where $\mathfrak{e}_{\mathcal{T}}$ denotes the function $Z \mapsto e^{2\pi i trace(\mathcal{T}\cdot Z)}$ on $\mathbb{H}_{2n}$.

We want to normalize the differential operators $\mathbb{D}$ in such a way that the

polynomial $\mathcal{P}$ has rational coefficients in all its components; this can always be done: Ibukiyama's differential operators can be chosen to have rational coefficients and then we have to divide by an appropriate power of $2\pi i$.
To handle the Fourier expansion of $\mathbb{D}E_k^{2n}$, it is convenient to use the standard basis

$$\mathfrak{a}_i := (0,\dots,0,1,0,\dots 0)^t \in V = \mathbb{C}^m \qquad (1 \leq i \leq m)$$

We may then write the polynomial $\mathcal{P}$ as a linear combination

$$\mathcal{P}(\mathcal{T}) = \sum_{i,j} \mathcal{P}_{i,j}(\mathcal{T}) \cdot \mathfrak{a}_i \otimes \mathfrak{a}_j$$

of scalar-valued polynomials $\mathcal{P}_{ij}$.

We consider the Fourier expansion of $\mathbb{D}E_k^{2n}$ as a function of w:

$$\mathbb{D}E^{2n}(z,w) = \sum_{T\in\Lambda^n} \sum_j \Phi_{T,j}(z) \otimes \mathfrak{a}_j e^{2\pi i trace(Tw)},$$

where $\Phi_{T,j}$ is an element of M_ρ^n, it is *cuspidal* because our differential operator maps modular forms to cusp forms, if the scalar weight is increased; this is why we imposed the condition $k' > k$. The Fourier expansion of any $\Phi_{T,j}$ can be written as

$$\Phi_{T,j}(z) = \sum_{S\in\Lambda^n} \sum_i \sum_R b_k^{2n}(\begin{pmatrix} S & R \\ R^t & S \end{pmatrix})\mathcal{P}_{i,j}(\begin{pmatrix} S & R \\ R^t & T \end{pmatrix})\mathfrak{a}_i e^{2\pi i trace(Sz)}$$

The summation over R goes over all matrices in $\frac{1}{2}\mathbb{Z}^{(n,n)}$, but due to the condition on positive-semidefiniteness, it is a finite sum. Clearly, the properties of the $b_k^{2n}(\mathcal{T})$ imply

$$\Phi_{T,j} \in S_\rho^n(\mathbb{Z})',$$

where the prime indicates that bounded denominators (i.e. bounded independent of $\mathcal{T}$) may occur.

3 Linearized pullback formula

Garrett [6] started to consider pullbacks of Eisenstein series. The version we use can be found in [1] for the scalar-valued case, the generalization to vector-valued cases is in [3]. The pullback formula then implies

$$\Phi_{T,j}(z) \sim \sum_{f_t} \frac{L(k-n, f_t)}{< f_t, f_t >} a_{f_t}^{(j)}(T) f_t(z). \tag{3}$$

Here f_t runs over an orthogonal basis of Hecke eigenforms in S_ρ^n, $<,>$ denotes the Petersson inner product of S_ρ^n and $L(s, f)$ is the standard L-function attached to f; we do not need the exact factor of proportionality here. Moreover, we have decomposed the Fourier coefficients $a_f(T)$ into its components:

$$a_f(T) = \sum_j a_f^{(j)}(T) \cdot \mathfrak{a}_j.$$

To give a linear version of (3), we consider the linear map

$$\Lambda : S_\rho^n \longrightarrow S_\rho^n,$$

defined by $f \longmapsto L(k-n, f) \cdot f$ for Hecke eigenforms. By the nonvanishing of $L(k-n, f)$, this defines an automorphism of S_ρ^n for $k > 2n$. Then (3) implies

$$< f, \Phi_{T,j} > \sim c_f^{(j)}(T) \tag{4}$$

where $\sum_T c_f(T) e^{2\pi i trace(TZ)}$ denotes the Fourier expansion of $\Lambda(f)$.
From (4) one can see that $\{\Phi_{T,j} \mid T \in \Lambda^n, 1 \leq j \leq m\}$ generates the full space S_ρ^n as a vector space and we have in this way confirmed (2), at least for cusp forms (of large weight).

Remark: In the above, we only covered a special situation: We should include non cusp forms, levels, low weights and half-integral weights. The method above, with some efforts, allows to get similar conclusions in those more general cases. It is however important to remark that while an analogue of (2) remains true for small weights, the stronger statement that vector-valued forms arise from scalar valued ones of higher degree by applying $\mathbb{D}$ may no longer hold in general !

Remark One can reformulate the results above using Jacobi forms instead of Siegel modular forms of degree $2n$: Every vector-valued Siegel modular form (cuspidal, of large weight, level one) arises as linear combination of $\mathbb{D}^J(\Phi)$ where Φ runs over scalar-valued Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{(n,n)}$ and $\mathbb{D}^J$ is an obvious Jacobi version of Ibukiymama's differential operator $\mathbb{D}$.

4 An application to congruences for vector-valued modular forms.

Here we report on our main motivation, which comes from p-adic vector-valued modular forms. In [2] we made some efforts to include vector-valued modular forms. In loc.cit. we left open, whether a vector-valued modular form for a congruence subgroup of type $\Gamma_0^n(p^m)$) with $m \geq 2$ gives rise to a p-adic modular form. (Note that the case $m = 1$ is different and here the standard method also works for the vector-valued case, see [2]). The reason was that in the scalar-valued case the operator

$$f \longmapsto f^p \mid U(p)$$

plays a central role to decrease the level from $\Gamma_0^n(p^m)$ to $\Gamma_0^n(p^{m-1})$ for $m \geq 2$. In the vector valued case a natural substitute for taking a p-th power is to consider a symmetric p-power, which however changes the representation ρ to $Sym^p(\rho)$ with a much larger representation space. An attempt to proceed along these lines was described in [5] with an annoyingly complicated definition of p-adic vector valued modular forms. For a geometric approach to vector-valued Siegel modular forms see [10].
We show now how this trouble can be avoided using the methods from above (taking for granted that appropriate generalizations hold for congruence subgroups and noncuspidal forms):

Theorem: *For any $F \in M_\rho^n(\Gamma_0(p^m))$ with $m \geq 2$ and all $N \geq 1$ there exists $k' > k$ and $\hat{F} \in M_{\rho'}^n(\Gamma_0(p^{m-1}))$ with $\rho' = \rho_0 \otimes \det^{k'}$ such that the congruence*

$$F \equiv \hat{F} \mod p^N$$

holds. All vector-valued Siegel modular forms for $\Gamma_0^n(p^m)$ are p-adic modular forms

Proof: After multiplication with a modular form congruent to 1 mod p^N we may assume that the weight of F is large. By the procedure of the previous session, we may assume that there is a scalar-valued $G \in M_l^{2n}(\Gamma_0(p^m))$ such that F can be expressed by finitely many $\Phi_{T,j}$ arising from $\mathbb{D}G(z,w)$. We may change G modulo $p^{N'}$ to a (again scalar-valued) $\hat{G}$ of level $\Gamma_0(p^{m-1})$. Then we use $\mathbb{D}\hat{G}(z,w)$.
The main point is to do the change of level *before* using the differential

operators. Possibly N' has to be chosen larger than N because the differential operator $\mathbb{D}$ may have powers of p in its denominator and the expression of F as a linear combination of the $\Phi_{T,j}$ may be not p-integral.

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